

# ON SOME DISCRETE MODEL OF THE MAGNETIC LAPLACIAN

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**ABSTRACT.** We construct some intrinsically defined discrete model of the magnetic Laplacian. The existence and uniqueness of solutions of the Dirichlet problem for the difference Poisson type equation are proved. We study in detail properties of the discrete model including the limiting process in the two-dimensional Euclidean case.

## 1. INTRODUCTION

Let  $(M, g)$  be a Riemannian manifold with a Riemannian metric  $(g_{ij})$  and  $\dim M = n$ . Denote  $\Lambda^p(M)$  the set of all differentiable complex-valued  $p$ -forms on  $M$  for each  $p = 0, 1, \dots, n$ . Note that  $\Lambda^0(M)$  is just  $C^\infty(M)$ . Denote also  $\Lambda_{(k)}^p(M)$  the set of all  $k$ -smooth (of the class  $C^k$ ) complex-valued  $p$ -forms on  $M$ . We define a *magnetic potential* as a real-valued 1-form  $A \in \Lambda_{(1)}^1(M)$ . So in local coordinates  $x_1, \dots, x_n$  it can be written as

$$A = \sum_{j=1}^n A_j dx^j,$$

where  $A_j = A_j(x)$  are real-valued functions of the class  $C^1$ .

Let  $*$  be the metric adjoint operator (Hodge star operator)  $*$  :  $\Lambda^p(M) \rightarrow \Lambda^{n-p}(M)$ . Then the invariant inner product of  $p$ -forms with compact support can be defined in a standard way by the relation

$$(1.1) \quad (\varphi, \psi) = \int_M \varphi \wedge * \bar{\psi},$$

where the bar over  $\psi$  means the complex conjugation and  $\wedge$  is the exterior multiplication operation. It is known that using the inner product (1.1) in spaces of smooth  $p$ -forms with compact support we can define the completions of these spaces. We denote these Hilbert spaces by  $L^2(M)$  for 0-forms (functions) and by  $L^2\Lambda^p(M)$  for  $p$ -forms,  $p = 1, \dots, n$ . Let us define the operator

$$d : L^2\Lambda^{p-1}(M) \rightarrow L^2\Lambda^p(M)$$

as the closure in the  $L^2$ -norm generated by the inner product (1.1) of the corresponding operator specified on smooth forms, i.e. as a strong extensions of the differential operator  $d : \Lambda^{p-1}(M) \rightarrow \Lambda^p(M)$ . We will need a deformed differential

$$(1.2) \quad d_A : L^2(M) \rightarrow L^2\Lambda^1(M), \quad \varphi \rightarrow d\varphi + i\varphi A,$$

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*Date:* October 2004.

1991 *Mathematics Subject Classification.* 35Q60, 39A12, 39A70.

*Key words and phrases.* magnetic Laplacian, discrete operators, difference equations.

where  $i = \sqrt{-1}$  and  $A$  is the magnetic potential.

The definition of the invariant inner product immediately induces the formal adjoint operator to the differential operator  $d_A$ . So we have the operator

$$\delta_A : L^2\Lambda^1(M) \rightarrow L^2(M)$$

giving by the relation

$$(d_A\varphi, \omega) = (\varphi, \delta_A\omega), \quad \varphi \in L^2(M), \quad \omega \in L^2\Lambda^1(M).$$

Here we assume that one of the forms have compact support. Then we can define the *magnetic Laplacian* as follows

$$(1.3) \quad -\Delta_A \equiv \delta_A d_A : L^2(M) \rightarrow L^2(M).$$

Let us identify the magnetic potential  $A$  with the multiplication operation

$$(1.4) \quad A : L^2(M) \rightarrow L^2\Lambda^1(M), \quad \varphi \rightarrow \varphi A$$

(see e.g. [7]). Then the formally adjoint operator  $\delta_A$  can be written as follows

$$(1.5) \quad \delta_A\omega = (\delta - iA^*)\omega,$$

where  $\delta$ ,  $A^*$  are the formal adjoint operators to  $d$  and  $A$  respectively. Using (1.2), (1.3), we can rewrite the magnetic Laplacian  $\Delta_A$  as follows

$$\begin{aligned} -\Delta_A\varphi &= (\delta - iA^*)(d\varphi + iA\varphi) \\ &= \delta d\varphi - iA^*d\varphi + i\delta(A\varphi) + A^*A\varphi \\ &= -\Delta\varphi - iA^*d\varphi + i\delta(A\varphi) + A^*A\varphi. \end{aligned}$$

Operator (1.3) is essentially self-adjoint (see for details [7, Th. 6.1]).

The main purpose of this paper is to construct an intrinsically defined discrete model of the magnetic Laplacian. Speaking about this discrete model we do not mean just the corresponding difference operator on a lattice or on graphs but we mean a discrete analog of the Riemannian structure on some combinatorial object. We consider discrete forms as certain cochains. We construct discrete analogs of the exterior multiplication operation, the Hodge star operator, of inner product (1.1) and the operators (1.2), (1.5).

Our approach bases on the formalism proposed by Dezin [3]. For an account of other geometric finite-difference approaches to Hodge theory of harmonic forms see references [1, 4, 6]. The discrete magnetic Laplacian on graphs had been studied in [2], [5].

In the spirit of [3], [8], [9] we study self-adjointness of the discrete magnetic Laplacian and we proof that the Dirichlet problem for the discrete Poisson type equation has a unique solution.

In this paper we consider just the two dimensional Euclidean case. Although similar constructions can be carried out in the  $n$ -dimensional case, the two-dimensional discrete model makes it possible to analyze in detail the combinatorial relations and the limiting process. One of the formal results is the construction of a nonstandard approximation of the generalized solution of the Poisson type equation for the magnetic Laplacian (1.3) under the minimal requirements of smoothness of the right hand side (is belonged to  $L^2$ ).

## 2. PRELIMINARIES ON COMBINATORIAL STRUCTURES

We use the schema of discretization due to Dezin [3]. Let  $\{x_k\}, \{e_k\}, k \in \mathbb{Z}$ , be the sets of basis elements of real linear spaces  $C^0, C^1$ . We will regard the linear combinations  $a = \sum a^k x_k, b = \sum b^k x_k, a^k, b^k \in \mathbb{R}$ , as zero-dimensional and one-dimensional chains, respectively.

It is convenient to introduce shift operators

$$\tau k = k + 1, \quad \sigma k = k - 1$$

in the set of indices. We define the one-dimensional complex  $C$  as the direct sum  $C^0 \oplus C^1$  of the introduced spaces with the following boundary operator

$$\partial x_k = 0, \quad \partial e_k = x_{\tau k} - x_k, \quad k \in \mathbb{Z}.$$

The definition of  $\partial$  is linearly extended to arbitrary chains. We call the complex  $C$  a combinatorial model of the real line. The basis elements  $x_k, e_k$  can be interpreted as points and intervals connecting the points (i.e.  $e_k = (x_k, x_{\tau k})$ ) of real line.

We consider the tensor degree  $C(n) = \otimes_1^n C$  of the one-dimensional complex  $C$  as a combinatorial model of  $\mathbb{R}^n$ . The main object of our study will be a discrete model of the magnetic Laplacian in the simplest two-dimensional domain. Therefore we describe the combinatorial relations that are encountered in the two-dimensional case.

The basis elements of the two-dimensional complex  $C(2)$  can be written as follows

$$\begin{aligned} x_k \otimes x_s &= x_{k,s}, & e_k \otimes x_s &= e_{k,s}^1, \\ e_k \otimes e_s &= V_{k,s}, & x_k \otimes e_s &= e_{k,s}^2. \end{aligned}$$

The boundary operator  $\partial$  we define as

$$\begin{aligned} \partial x_{k,s} &= 0, & \partial e_{k,s}^1 &= x_{\tau k,s} - x_{k,s}, & \partial e_{k,s}^2 &= x_{k,\tau s} - x_{k,s}, \\ \partial V_{k,s} &= e_{k,s}^1 + e_{\tau k,s}^2 - e_{k,\tau s}^1 - e_{k,s}^2. \end{aligned} \quad (2.1)$$

Let us introduce an object dual to  $C(2)$ . Namely, the complex of complex-valued functions over  $C(2)$ . The dual complex  $K(2)$  we can consider as the set of complex-valued cochains and it has the same structure as  $C(2)$ , i.e.  $K(2) = K \otimes K$ . In other words,  $K(2)$  is a linear complex space with basis elements  $\{x^{k,s}, e_1^{k,s}, e_2^{k,s}, V^{k,s}\}$ .

The pairing (chain-cochain) operation is defined by the rules:

$$(2.2) \quad \langle x_{k,s}, x^{p,q} \rangle = \langle e_{k,s}^1, e_1^{p,q} \rangle = \langle e_{k,s}^2, e_2^{p,q} \rangle = \langle V_{k,s}, V^{p,q} \rangle = \delta_{k,s}^{p,q},$$

where  $\delta_{k,s}^{p,q}$  is Kronecker symbol. We call elements of the complex  $K(2)$  forms. Then the 0-, 1-, 2-forms  $\varphi, \omega = (u, v), \eta$  can be written as

$$(2.3) \quad \varphi = \sum_{k,s} \varphi_{k,s} x^{k,s}, \quad \omega = \sum_{k,s} (u_{k,s} e_1^{k,s} + v_{k,s} e_2^{k,s}), \quad \eta = \sum_{k,s} \eta_{k,s} V^{k,s},$$

where  $\varphi_{k,s}, u_{k,s}, v_{k,s}, \eta_{k,s} \in \mathbb{C}$  for any  $k, s \in \mathbb{Z}$ . The pairing (2.2) is linearly extended to forms (2.3). The boundary operation  $\partial$  in  $C(2)$  (2.1) induces the dual operation  $d^c$  in  $K(2)$ :

$$(2.4) \quad \langle \partial a, \alpha \rangle = \langle a, d^c \alpha \rangle,$$

where  $a \in C(2), \alpha \in K(2)$ . The coboundary operator  $d^c$  is a discrete analog of the exterior differentiation operator  $d$ . We will need the expression for  $d^c$  over the

basis elements of  $C(2)$ :

$$(2.5) \quad \begin{aligned} &< e_{k,s}^1, d^c \varphi > = \varphi_{\tau k,s} - \varphi_{k,s} \equiv \Delta_k \varphi_{k,s}, \\ &< e_{k,s}^2, d^c \varphi > = \varphi_{k,\tau s} - \varphi_{k,s} \equiv \Delta_s \varphi_{k,s}, \\ &< V_{k,s}, d^c \omega > = v_{\tau k,s} - v_{k,s} - u_{k,\tau s} + u_{k,s} \equiv \Delta_k v_{k,s} - \Delta_s u_{k,s}. \end{aligned}$$

We define the multiplication  $\cup$  in  $K(2)$  by the rules:

$$(2.6) \quad \begin{aligned} x^{k,s} \cup x^{k,s} &= x^{k,s}, & e_2^{k,s} \cup e_1^{k,\tau s} &= -V^{k,s}, \\ x^{k,s} \cup e_1^{k,s} &= e_1^{k,s} \cup x^{\tau k,s} = e_1^{k,s}, & x^{k,s} \cup e_2^{k,s} &= e_2^{k,s} \cup x^{k,\tau s} = e_2^{k,s}, \\ x^{k,s} \cup V^{k,s} &= V^{k,s} \cup x^{\tau k,\tau s} = e_1^{k,s} \cup e_2^{\tau k,s} = V^{k,s}, \end{aligned}$$

supposing the product to be zero in all other cases. To forms (2.3) the  $\cup$ -multiplication can be extended linearly. For arbitrary forms  $\alpha, \beta \in K(2)$  we have (see [3, p. 147]) the relation

$$(2.7) \quad d^c(\alpha \cup \beta) = d^c \alpha \cup \beta + (-1)^r \alpha \cup d^c \beta,$$

where  $r$  is the dimension of the form  $\alpha$ . So the  $\cup$ -multiplication is an analog of the exterior multiplication  $\wedge$  for differential forms.

Let  $\varepsilon^{k,s}$  be an arbitrary basis elements of  $K(2)$ . We introduce the "star" operator setting

$$(2.8) \quad \varepsilon^{k,s} \cup * \varepsilon^{k,s} = V^{k,s}.$$

Using (2.6), we have

$$*x^{k,s} = V^{k,s}, \quad *e_1^{k,s} = e_2^{\tau k,s}, \quad *e_2^{k,s} = -e_1^{k,\tau s}, \quad *V^{k,s} = x^{\tau k,\tau s}.$$

The operator  $*$  is extended to arbitrary forms by linearity.

Let now

$$(2.9) \quad V = \sum_{k,s} V_{k,s}, \quad k = 1, 2, \dots, N, \quad s = 1, 2, \dots, M$$

is some fixed "domain", namely, a set of 2-dimensional basis elements of  $C(2)$ . Then the relation

$$(2.10) \quad (\alpha, \beta)_V = \langle V, \alpha \cup * \bar{\beta} \rangle$$

gives a correct definition of the inner product for forms of the same degree (cf. (1.1)). For forms of different degree the product (2.10) is equal to zero. Using (2.6)–(2.9), we obtain

$$(2.11) \quad (\alpha, \beta)_V = \sum_{k,s} \alpha_{k,s} \bar{\beta}_{k,s},$$

where  $\alpha_{k,s}, \beta_{k,s}$  are components of the forms  $\alpha, \beta \in K(2)$ .

We agree that in what follows, unless the limits of summation are specified, the subscripts  $k, s$  always run over the set of values indicated in (2.9).

Taking into account (2.7), (2.10), we can written for a  $(p-1)$ -form  $\alpha \in K(2)$  and  $p$ -form  $\beta \in K(2)$  the relation

$$(2.12) \quad (d^c \alpha, \beta)_V = \langle \partial V, \alpha \cup * \bar{\beta} \rangle + (\alpha, \delta^c \beta)_V,$$

where

$$(2.13) \quad \delta^c \beta = (-1)^p *^{-1} d^c * \beta.$$

Here  $*^{-1}$  is the operation inverse to  $*$ , i.e.  $*^{-1}* = 1$ . If the form  $\alpha \cup *\bar{\beta}$  vanishes on the boundary  $\partial V$ , then Equation (2.13) defines the formally adjoint operator to  $d^c$ . Let  $\omega = (u, v)$  be an 1-form (2.3). Then we have

$$(2.14) \quad \delta^c \omega = \sum_{k,s} (-\Delta_k u_{\sigma k,s} - \Delta_s v_{k,\sigma s}) x^{k,s}.$$

We call the operator  $\delta^c$  a discrete analog of the codifferential  $\delta$ .

Therefore a discrete analog of the Laplace operator can be defined as follows

$$-\Delta^c = \delta^c d^c + d^c \delta^c.$$

If  $\varphi$  is a 0-form, then  $-\Delta^c \varphi = \delta^c d^c \varphi$  and we obtain at the point  $x_{k,s}$  the difference expression

$$\langle x_{k,s}, -\Delta^c \varphi \rangle = 4\varphi_{k,s} - \varphi_{\tau k,s} - \varphi_{k,\tau s} - \varphi_{\sigma k,s} - \varphi_{k,\sigma s}.$$

It should be noted that the definition of the inner product (2.11) turns the linear space of forms over  $V$  into finite-dimensional Hilbert spaces  $H^0, H^1, H^2$  with bases  $\{x^{k,s}\}, \{e_1^{k,s}, e_2^{k,s}\}, \{V^{k,s}\}, k = 1, 2, \dots, N, s = 1, 2, \dots, M$ , respectively. Thus we can regard the operators  $d^c, \delta^c, \Delta^c$  over  $V$  as follow

$$d^c : H^p \rightarrow H^{p+1}, \quad \delta^c : H^p \rightarrow H^{p-1}, \quad \Delta^c : H^p \rightarrow H^p,$$

where  $p = 0, 1, 2$ . It is convenient to suppose that  $H^{-1} = H^3 = 0$ .

### 3. DISCRETE MODEL OF THE MAGNETIC LAPLACIAN

Let a real-valued 1-form

$$A = \sum_{k,s} (A_{k,s}^1 e_1^{k,s} + A_{k,s}^2 e_2^{k,s}),$$

$A_{k,s}^1, A_{k,s}^2 \in \mathbb{R}$ , be a discrete analog of the *magnetic potential*. We define the discrete analog of the deformed differential (1.2) as follows

$$(3.1) \quad d_A^c : H^0 \rightarrow H^1, \quad \varphi \rightarrow d^c \varphi + i\varphi \cup A.$$

Taking into account (2.5), (2.6), we have

$$(3.2) \quad d_A^c \varphi = \sum_{k,s} ((\Delta_k \varphi_{k,s} + i\varphi_{k,s} A_{k,s}^1) e_1^{k,s} + (\Delta_s \varphi_{k,s} + i\varphi_{k,s} A_{k,s}^2) e_2^{k,s}).$$

As in the continual case (see (1.4)), we can identify the discrete magnetic potential  $A$  with the multiplication operator

$$(3.3) \quad A : H^0 \rightarrow H^1, \quad \varphi \rightarrow \varphi \cup A.$$

Then we have

$$A\varphi = \sum_{k,s} (\varphi_{k,s} A_{k,s}^1 e_1^{k,s} + \varphi_{k,s} A_{k,s}^2 e_2^{k,s}).$$

**Proposition 3.1.** *The formally adjoint operator  $A^* : H^1 \rightarrow H^0$  acts on an arbitrary 1-form  $\omega = (u, v)$  as follows*

$$(3.4) \quad A^* \omega = \sum_{k,s} (A_{k,s}^1 u_{k,s} + A_{k,s}^2 v_{k,s}) x^{k,s}.$$

*Proof.* Since the 1-form  $A \in H^1$  is real-valued by assumption, we have

$$\begin{aligned} (A\varphi, \omega)_V &= (\varphi \cup A, \omega)_V = \langle V, (\varphi \cup A) \cup *\bar{\omega} \rangle \\ &= \sum_{k,s} ((\varphi_{k,s} A_{k,s}^1) \overline{u_{k,s}} + (\varphi_{k,s} A_{k,s}^2) \overline{v_{k,s}}) \\ &= \sum_{k,s} \varphi_{k,s} (\overline{A_{k,s}^1 u_{k,s} + A_{k,s}^2 v_{k,s}}) = (\varphi, A^* \omega)_V. \end{aligned}$$

□

Let us suppose that components  $\alpha_{k,s}$  of an arbitrary  $r$ -form  $\alpha \in H^r$ ,  $r = 0, 1, 2$ , satisfy the following "boundary conditions":

$$(3.5) \quad \alpha_{0,s} = \alpha_{\tau N, s} = 0, \quad \alpha_{k,0} = \alpha_{k,\tau M} = 0$$

for all  $k = 1, 2, \dots, N$ ,  $s = 1, 2, \dots, M$ .

**Proposition 3.2.** *Let components of  $\varphi \in H^0$ ,  $\omega \in H^1$  satisfy Conditions (3.5). Then*

$$(d_A^c \varphi, \omega)_V = (\varphi, \delta_A^c \omega)_V,$$

where

$$(3.6) \quad \delta_A^c \omega = \delta^c \omega - iA^* \omega.$$

*Proof.* Note that Conditions (3.5) imply the relation  $\langle \partial V, \varphi \cup *\bar{\omega} \rangle = 0$  [3, p. 161]. Then from (2.12) we have

$$(d^c \varphi, \omega)_V = (\varphi, \delta^c \omega)_V.$$

Hence

$$\begin{aligned} (d_A^c \varphi, \omega)_V &= (d^c \varphi + i\varphi \cup A, \omega)_V = (d^c \varphi, \omega)_V + i(\varphi \cup A, \omega)_V \\ &= (\varphi, \delta^c \omega)_V + i(\varphi, A^* \omega)_V = (\varphi, (\delta^c - iA^*)\omega)_V. \end{aligned}$$

□

Thus the operator  $\delta_A^c : H^1 \rightarrow H^0$  is the formally adjoint operator to the operator  $d_A^c$ . Using (2.14), we can rewrite (3.6) in a "pointwise" form:

$$\langle x_{k,s}, \delta_A^c \omega \rangle = -\Delta_k u_{\sigma k, s} - \Delta_s v_{k, \sigma s} - i(A_{k,s}^1 u_{k,s} + A_{k,s}^2 v_{k,s}).$$

We have

$$\begin{aligned} \varphi \cup \delta^c \omega &= \sum_{k,s} \varphi_{k,s} (-\Delta_k u_{\sigma k, s} - \Delta_s v_{k, \sigma s}) x^{k,s} \\ &= \sum_{k,s} (\varphi_{\sigma k, s} u_{\sigma k, s} - \varphi_{k, s} u_{k, s} - \varphi_{k, s} v_{k, s} + \varphi_{k, \sigma s} v_{k, \sigma s}) x^{k,s} \\ &\quad + \sum_{k,s} (\varphi_{k, s} u_{\sigma k, s} - \varphi_{\sigma k, s} u_{\sigma k, s} + \varphi_{k, s} v_{k, \sigma s} - \varphi_{k, \sigma s} v_{k, \sigma s}) x^{k,s} \\ &= \delta^c (\varphi \cup \omega) + \sum_{k,s} ((\Delta_k \varphi_{\sigma k, s}) u_{\sigma k, s} + (\Delta_s \varphi_{k, \sigma s}) v_{k, \sigma s}) x^{k,s}. \end{aligned}$$

It follows that

$$\delta^c (\varphi \cup \omega) = \varphi \cup \delta^c \omega - \sum_{k,s} ((\Delta_k \varphi_{\sigma k, s}) u_{\sigma k, s} + (\Delta_s \varphi_{k, \sigma s}) v_{k, \sigma s}) x^{k,s}.$$

From this we immediately obtain the following discrete Leibniz rule for  $\delta_A^c$ :

$$\begin{aligned}\delta_A^c(\varphi \cup \omega) &= (\delta^c - iA^*)(\varphi \cup \omega) = \delta^c(\varphi \cup \omega) - iA^*(\varphi \cup \omega) \\ &= \varphi \cup \delta^c \omega - \varphi \cup iA^* \omega - \sum_{k,s} ((\Delta_k \varphi_{\sigma k,s}) u_{\sigma k,s} + (\Delta_s \varphi_{k,\sigma s}) v_{k,\sigma s}) x^{k,s} \\ &= \varphi \cup \delta_A^c \omega - \sum_{k,s} ((\Delta_k \varphi_{\sigma k,s}) u_{\sigma k,s} + (\Delta_s \varphi_{k,\sigma s}) v_{k,\sigma s}) x^{k,s}\end{aligned}$$

(cf. [7, Sect. 2], where the corresponding Leibniz rule is given in the continual case).

Let us define the discrete magnetic Laplacian as

$$-\Delta_A^c = \delta_A^c d_A^c : H^0 \rightarrow H^0.$$

Note that we assume that Conditions (3.5) are satisfied for any form  $\varphi \in H^0$ . This gives us the necessary extension of  $\varphi$  beyond  $H^0$  to consider the operator  $-\Delta_A^c$  as above.

Using (3.1), (3.6), we have

$$\begin{aligned}-\Delta_A^c \varphi &= \delta_A^c (d^c \varphi + i\varphi \cup A) \\ &= (\delta^c - iA^*) d^c \varphi + (\delta^c - iA^*) (i\varphi \cup A) \\ &= -\Delta^c \varphi - iA^* d^c \varphi + i\delta^c (\varphi \cup A) + A^* (\varphi \cup A) \\ (3.7) \quad &= -\Delta^c \varphi - iA^* d^c \varphi + i\delta^c A \varphi + A^* A \varphi.\end{aligned}$$

**Proposition 3.3.** *The operator  $-\Delta_A^c$  is self-adjoint, i. e.*

$$(\delta_A^c d_A^c \varphi, \psi)_V = (\varphi, \delta_A^c d_A^c \psi)_V.$$

*Proof.* It is known (see [3, p. 163]) that under Conditions (3.5) the discrete Laplacian  $-\Delta^c = \delta^c d^c : H^0 \rightarrow H^0$  is self-adjoint. Using Propositions 3.1, 3.2, we obtain

$$\begin{aligned}(\delta_A^c d_A^c \varphi, \psi)_V &= (\delta^c d^c \varphi, \psi)_V - (iA^* d^c \varphi, \psi)_V + (i\delta^c A \varphi, \psi)_V + (A^* A \varphi, \psi)_V \\ &= (\varphi, \delta^c d^c \psi)_V + (d^c \varphi, iA \psi)_V - (A \varphi, i d^c \psi)_V + (A \varphi, A \psi)_V \\ &= (\varphi, -\Delta^c \psi)_V + (\varphi, i\delta^c A \psi)_V - (\varphi, iA^* d^c \psi)_V + (\varphi, A^* A \psi)_V \\ &= (\varphi, (-\Delta^c + i\delta^c A - iA^* d^c + A^* A) \psi)_V = (\varphi, -\Delta_A^c \psi)_V.\end{aligned}$$

□

Using (3.1), we can write

$$\begin{aligned}(3.8) \quad (d_A^c \varphi, d_A^c \psi)_V &= (d^c \varphi, d^c \psi)_V + (d^c \varphi, iA \psi)_V \\ &\quad + (iA \varphi, d^c \psi)_V + (iA \varphi, iA \psi)_V.\end{aligned}$$

Taking into account (2.5) and (2.11), we have

$$\begin{aligned}
(d^c \varphi, A\psi)_V &= \sum_{k,s} (\Delta_k \varphi_{k,s} (\overline{\psi_{k,s}} A_{k,s}^1) + \Delta_s \varphi_{k,s} (\overline{\psi_{k,s}} A_{k,s}^2)) \\
&= \sum_{k,s} \varphi_{k,s} (-\Delta_k (\overline{\psi_{\sigma k,s}} A_{\sigma k,s}^1) - \Delta_s (\overline{\psi_{k,\sigma s}} A_{k,\sigma s}^2)) \\
&+ \sum_s (\varphi_{\tau N,s} (\overline{\psi_{N,s}} A_{N,s}^1) - \varphi_{1,s} (\overline{\psi_{0,s}} A_{0,s}^1)) \\
&+ \sum_k (\varphi_{k,\tau M} (\overline{\psi_{k,M}} A_{k,M}^2) - \varphi_{k,1} (\overline{\psi_{k,0}} A_{k,0}^2)) \\
&= \sum_s (\varphi_{\tau N,s} (\overline{\psi_{N,s}} A_{N,s}^1) - \varphi_{1,s} (\overline{\psi_{0,s}} A_{0,s}^1)) \\
&+ \sum_k (\varphi_{k,\tau M} (\overline{\psi_{k,M}} A_{k,M}^2) - \varphi_{k,1} (\overline{\psi_{k,0}} A_{k,0}^2)) + (\varphi, \delta^c A\psi)_V.
\end{aligned}$$

It follows that

$$\begin{aligned}
(d_A^c \varphi, d_A^c \psi)_V &= \sum_s [\varphi_{\tau N,s} (\overline{\psi_{\tau N,s}} - \overline{\psi_{N,s}} - i\overline{\psi_{N,s}} A_{N,s}^1)] \\
&- \sum_s [\varphi_{1,s} (\overline{\psi_{1,s}} - \overline{\psi_{0,s}} + i\overline{\psi_{0,s}} A_{0,s}^1)] \\
&+ \sum_k [\varphi_{k,\tau M} (\overline{\psi_{k,\tau M}} - \overline{\psi_{k,M}} - i\overline{\psi_{k,M}} A_{k,M}^2)] \\
&- \sum_k [\varphi_{k,1} (\overline{\psi_{k,1}} - \overline{\psi_{k,0}} + i\overline{\psi_{k,0}} A_{k,0}^2)] \\
&+ (\varphi, -\Delta^c \psi)_V + (\varphi, i\delta^c A\psi)_V - (\varphi, iA^*(d^c \psi))_V + (\varphi, A^* A\psi)_V.
\end{aligned}$$

Consequently, if the forms  $\varphi, \psi \in H^0$  satisfy Conditions (3.5), then we obtain

$$(3.9) \quad (d_A^c \varphi, d_A^c \psi)_V = - \sum_s \varphi_{1,s} \overline{\psi_{1,s}} - \sum_k \varphi_{k,1} \overline{\psi_{k,1}} + (\varphi, \delta_A^c d_A^c \psi)_V.$$

**Theorem 3.4.** *For any form  $f \in H^0$  a solution of the equation*

$$(3.10) \quad -\Delta_A^c \varphi = f$$

*exists and is unique.*

*Proof.* By virtue of the self-adjointness of the operator  $-\Delta_A^c$  it is enough to prove the uniqueness of the solution. Assume that  $\varphi = \psi$  in Equation (3.9). Then we can write

$$(3.11) \quad (d_A^c \varphi, d_A^c \varphi)_V + \sum_s |\varphi_{1,s}|^2 + \sum_k |\varphi_{k,1}|^2 = (\varphi, -\Delta_A^c \varphi)_V.$$

Using (2.5), (2.11), we get

$$(d^c \varphi, d^c \varphi)_V = \sum_{k,s} (|\Delta_k \varphi_{k,s}|^2 + |\Delta_s \varphi_{k,s}|^2),$$

$$\begin{aligned}
(d^c \varphi, iA\varphi)_V + (iA\varphi, d^c \varphi)_V &= i \sum_{k,s} [A_{k,s}^1 (\varphi_{k,s} (\overline{\Delta_k \varphi_{k,s}}) - (\Delta_k \varphi_{k,s}) \overline{\varphi_{k,s}})] \\
&\quad + i \sum_{k,s} [A_{k,s}^2 (\varphi_{k,s} (\overline{\Delta_s \varphi_{k,s}}) - (\Delta_s \varphi_{k,s}) \overline{\varphi_{k,s}})], \\
(iA\varphi, iA\varphi)_V &= \sum_{k,s} ((A_{k,s}^1)^2 |\varphi_{k,s}|^2 + (A_{k,s}^2)^2 |\varphi_{k,s}|^2).
\end{aligned}$$

It is easy to check that

$$\varphi_{k,s} (\overline{\Delta_k \varphi_{k,s}}) - (\Delta_k \varphi_{k,s}) \overline{\varphi_{k,s}} = 2i (\operatorname{Im}(\varphi_{k,s}) \operatorname{Re}(\Delta_k \varphi_{k,s}) - \operatorname{Re}(\varphi_{k,s}) \operatorname{Im}(\Delta_k \varphi_{k,s})).$$

Substituting the last relations into (3.8) we obtain

$$\begin{aligned}
(d_A^c \varphi, d_A^c \varphi)_V &= \sum_{k,s} \left[ |\Delta_k \varphi_{k,s}|^2 + |\Delta_s \varphi_{k,s}|^2 + (A_{k,s}^1)^2 |\varphi_{k,s}|^2 + (A_{k,s}^2)^2 |\varphi_{k,s}|^2 \right. \\
&\quad + 2A_{k,s}^1 \operatorname{Re}(\varphi_{k,s}) \operatorname{Im}(\Delta_k \varphi_{k,s}) - 2A_{k,s}^1 \operatorname{Im}(\varphi_{k,s}) \operatorname{Re}(\Delta_k \varphi_{k,s}) \\
&\quad \left. + 2A_{k,s}^2 \operatorname{Re}(\varphi_{k,s}) \operatorname{Im}(\Delta_s \varphi_{k,s}) - 2A_{k,s}^2 \operatorname{Im}(\varphi_{k,s}) \operatorname{Re}(\Delta_s \varphi_{k,s}) \right] \\
&= \sum_{k,s} \left[ \left( \operatorname{Re}(\Delta_k \varphi_{k,s}) - A_{k,s}^1 \operatorname{Im}(\varphi_{k,s}) \right)^2 + \left( \operatorname{Im}(\Delta_k \varphi_{k,s}) + A_{k,s}^1 \operatorname{Re}(\varphi_{k,s}) \right)^2 \right. \\
&\quad \left. + \left( \operatorname{Re}(\Delta_s \varphi_{k,s}) - A_{k,s}^2 \operatorname{Im}(\varphi_{k,s}) \right)^2 + \left( \operatorname{Im}(\Delta_s \varphi_{k,s}) + A_{k,s}^2 \operatorname{Re}(\varphi_{k,s}) \right)^2 \right].
\end{aligned}$$

Now let we take  $f = 0$  in Equation (3.10). Then comparing the last equation and (3.11), we obtain

$$\begin{aligned}
&\sum_{k,s} \left[ \left( \operatorname{Re}(\Delta_k \varphi_{k,s}) - A_{k,s}^1 \operatorname{Im}(\varphi_{k,s}) \right)^2 + \left( \operatorname{Im}(\Delta_k \varphi_{k,s}) + A_{k,s}^1 \operatorname{Re}(\varphi_{k,s}) \right)^2 \right. \\
&\quad \left. + \left( \operatorname{Re}(\Delta_s \varphi_{k,s}) - A_{k,s}^2 \operatorname{Im}(\varphi_{k,s}) \right)^2 + \left( \operatorname{Im}(\Delta_s \varphi_{k,s}) + A_{k,s}^2 \operatorname{Re}(\varphi_{k,s}) \right)^2 \right] \\
&+ \sum_s |\varphi_{1,s}|^2 + \sum_k |\varphi_{k,1}|^2 = 0.
\end{aligned}$$

It follows that  $\varphi_{k,s} = 0$  for any  $k, s$ . Hence  $\varphi \equiv 0$ .  $\square$

This immediately implies the following statement:

**Corollary 3.5.** *The operator  $-\Delta_A^c$  is positiv.*

#### 4. APPROXIMATION AND LIMITING PROCESS

In this section we consider the relationship between the combinatorial objects that we have described above and the corresponding continual objects. We will construct some nonstandard approximation of the generalized solution of the Poisson type equation

$$(4.1) \quad -\Delta_A \varphi = f,$$

where  $f \in L^2(\Omega)$ . We will realize the scheme similar to that given in [3, Ch.3, Sec.3].

Let the domain  $\Omega \in \mathbb{R}^2$  be a rectangle with vertices  $(a_1, b_1)$ ,  $(a_2, b_1)$ ,  $(a_1, b_2)$ ,  $(a_2, b_2)$ , where  $0 \leq a_1 < a_2$ ,  $0 \leq b_1 < b_2$ . We introduce a scale  $h$  setting  $h = N^{-1}(a_2 - a_1) = M^{-1}(b_2 - b_1)$ . Divide  $\Omega$  by the following straight lines

$$x = a_1 + kh, \quad y = b_1 + sh, \quad k = 0, 1, \dots, N, \quad s = 0, 1, \dots, M.$$

We denote by  $x_{k,s}$  the point of intersection of these lines. We denote by  $V_{k,s}$  an open square bounded by the lines:  $x = a_1 + kh$ ,  $y = b_1 + sh$ ,  $x = a_1 + \tau kh$ ,  $y = b_1 + \tau sh$ . Let  $e_{k,s}^1$  and  $e_{k,s}^2$  be the horizontal and vertical sides of  $V_{k,s}$ , i. e.  $e_{k,s}^1 = (x_{k,s}, x_{\tau k, s})$ ,  $e_{k,s}^2 = (x_{k,s}, x_{k, \tau s})$ . In this way we identify the rectangle  $\Omega$  with the combinatorial domain  $V$  (2.9).

Let us now compare every discrete form  $\varphi \in H^0$  with the step function assuming that

$$\varphi^h(x, y) = \varphi_{k,s}, \quad \text{for } x, y \in V_{k,s}.$$

In the case of the 1-form  $\omega = (u, v) \in H^1$  we have the pair of step functions  $u^h(x, y) = u_{k,s}$ ,  $v^h(x, y) = v_{k,s}$  and we can write  $\omega^h(x, y) = u^h(x, y)dx + v^h(x, y)dy$ . Recall that  $\varphi_{k,s}, u_{k,s}, v_{k,s} \in \mathbb{C}$  for any  $k, s$ . Then  $\varphi^h, \omega^h$  are complex-valued.

It is easy to check that

$$(4.2) \quad \|\varphi^h\|_{L^2(\Omega)} = h\|\varphi\|_{H^0}, \quad \|\omega^h\|_{L^2\Lambda^1(\Omega)} = h\|\omega\|_{H^1}.$$

Define difference operators acting on the step functions as follows

$$\begin{aligned} \Delta_x^h \varphi^h(x, y) &= h^{-1}(\varphi^h(x+h, y) - \varphi^h(x, y)), \\ \Delta_y^h \varphi^h(x, y) &= h^{-1}(\varphi^h(x, y+h) - \varphi^h(x, y)). \end{aligned}$$

Replacing the partial derivatives  $\frac{\partial}{\partial x}$ ,  $\frac{\partial}{\partial y}$  appearing in  $d$ ,  $\delta$  by the difference operators  $\Delta_x^h$ ,  $\Delta_y^h$ , we can introduce the difference operators  $d^h$ ,  $\delta^h$ . The difference equation

$$(4.3) \quad d^h \varphi^h = \omega^h$$

is equivalent to the following family of equations

$$\Delta_k \varphi_{k,s} = h u_{k,s}, \quad \Delta_s \varphi_{k,s} = h v_{k,s},$$

where  $k = 0, 1, \dots, N$ ,  $s = 0, 1, \dots, M$ . Hence Equation (4.3) can be rewritten as the following discrete equation

$$d^c \varphi = h\omega,$$

where  $\varphi$ ,  $\omega$  are discrete forms (see (2.3)) with the components  $\varphi_{k,s}$  and  $u_{k,s}, v_{k,s}$ , respectively. Similarly, we associate the difference equation  $\delta^h \omega^h = \varphi^h$  and the discrete equation  $\delta^c \omega = h\varphi$ .

We can also introduce the following difference operator

$$(4.4) \quad d_{A^h}^h = d^h + iA^h,$$

where  $A^h = A^{1h}dx + A^{2h}dy$  and  $A^{1h}, A^{2h}$  are real-valued step functions defined as above. On the other hand, we can consider the step 1-form  $A^h$  as the multiplication operator  $A^h : L^2(\Omega) \rightarrow L^2\Lambda^1(\Omega)$  acting as follows

$$A^h \varphi^h = \varphi^h A^{1h}dx + \varphi^h A^{2h}dy.$$

Then the formally adjoint operator to  $A^h$  (cf. (3.4)) acts on a step 1-form  $\omega^h = (u^h, v^h)$  as

$$(A^h)^* \omega^h = u^h A^{1h} + v^h A^{2h}.$$

Thus we define the difference magnetic Laplacian (cf. (3.7)) by the formula

$$(4.5) \quad -\Delta_{A^h}^h \equiv \delta_{A^h}^h d_{A^h}^h = \delta^h d^h - i(A^h)^* d^h + i\delta^h A^h + (A^h)^* A^h.$$

Now we consider the *discretization* procedure (see for details [3, p. 170]). Let  $f(x, y)$  be a complex-valued function defined over  $\Omega$  (or over  $V = \sum V_{k,s}$ ) and let  $f \in L^2(\Omega)$ . Associate  $f$  with the step function  $f^h$  setting

$$(4.6) \quad f^h(x, y) = h^{-2} \int_{V_{k,s}} f(\xi, \eta) d\xi d\eta, \quad \text{for } x, y \in V_{k,s}.$$

Moreover, the value of  $f^h$  can be assigned to the point  $x_{k,s}$ . As above, we can write  $f^h(x, y) = f_{k,s}$  for  $x, y \in V_{k,s}$ . Thus we obtain the discrete 0-form  $\widehat{f} = \sum f_{k,s} x^{k,s}$  which is associating with  $f \in L^2(\Omega)$ . Similarly, if  $f$  is an 1-form,  $f \in L^2\Lambda^1(\Omega)$ , then we associate each component of  $f$  with the step function (4.6) and we assign the value of it to one of the intervals  $e_{k,s}^1$  or  $e_{k,s}^2$ .

Let us introduce the norm

$$\|\varphi^h\|_{W(\Omega)}^2 = \int_{\Omega} (|\Delta_x^h \varphi^h|^2 + |\Delta_y^h \varphi^h|^2) dx dy.$$

It is not difficult to verify that

$$(4.7) \quad \|\varphi^h\|_{W(\Omega)}^2 = \sum_{k=0}^N \sum_{s=0}^M (|\Delta_k \varphi_{k,s}|^2 + |\Delta_s \varphi_{k,s}|^2).$$

Hence we can write

$$\|\varphi^h\|_{W(\Omega)} = \|\varphi\|_{W(V)}.$$

**Theorem 4.1.** *Let the step function  $f^h$  be the discretization of  $f \in L^2(\Omega)$ . Then the following Dirichlet problem*

$$(4.8) \quad -\Delta_{A^h}^h \varphi^h = f^h,$$

$$(4.9) \quad \varphi^h|_{\partial\Omega} = 0$$

*has a unique solution and the inequality*

$$(4.10) \quad \|\varphi^h\|_{W(\Omega)} < c_1 \|\operatorname{Re} f\|_{L^2(\Omega)} + c_2 \|\operatorname{Im} f\|_{L^2(\Omega)}$$

*is valid for the solution  $\varphi^h$ .*

*Proof.* Using (4.5), Equation (4.8) can be rewritten as

$$(4.11) \quad \delta^h d^h \varphi^h - i(A^h)^* d^h \varphi^h + i\delta^h A^h \varphi^h + (A^h)^* A^h \varphi^h = f^h.$$

By definition the step function  $\varphi^h$  and the step form  $A^h$  on  $\Omega$  associated with the discrete forms  $\varphi$  and  $A$  on  $V$ . As above, if we replace the difference operator  $d^h, \delta^h$  by the discrete operator  $d^c, \delta^c$ , then Equation (4.11) transforms into the following equation

$$(4.12) \quad \delta^c d^c \varphi - i h A^* d^c \varphi + i h \delta^c A \varphi + h^2 A^* A \varphi = h^2 \widehat{f},$$

where  $\widehat{f}$  is a 0-form defined by the step function  $f^h$  (see (4.6)). Note that, if the step function  $\varphi^h$  satisfies Condition (4.9), then the corresponding discrete form satisfies Condition (3.5). Thus the unique solvability of (4.8), (4.9) immediately follows from Theorem 3.4.

Let now represent Equation (4.8) as follows

$$-\Delta_{A^h}^h \operatorname{Re} \varphi^h = \operatorname{Re} f^h, \quad -\Delta_{A^h}^h \operatorname{Im} \varphi^h = \operatorname{Im} f^h.$$

In a similar way we can split Equation (4.12).

Since

$$(d^c \alpha, iA\alpha)_V + (iA\alpha, d^c \alpha)_V = 0$$

for any real-valued discrete form  $\alpha \in H^0$  it follows that from (3.7), (3.11) we obtain

$$\|d^c \operatorname{Re} \varphi\|_{H^1}^2 + \sum_s (\operatorname{Re} \varphi_{1,s})^2 + \sum_k (\operatorname{Re} \varphi_{k,1})^2 + h^2 \|A \operatorname{Re} \varphi\|_{H^1}^2 = h^2 (\operatorname{Re} \varphi, \operatorname{Re} \widehat{f})_V.$$

It immediately follows that

$$\|\operatorname{Re} \varphi\|_{W(V)}^2 < h^2 (\operatorname{Re} \varphi, \operatorname{Re} \widehat{f})_V = h^2 \sum_{k,s} \operatorname{Re} \varphi_{k,s} \operatorname{Re} f_{k,s}.$$

Hence, using (4.2), we have

$$(4.13) \quad \|\operatorname{Re} \varphi^h\|_{W(\Omega)}^2 < \|\operatorname{Re} \varphi^h\|_{L^2(\Omega)} \|\operatorname{Re} f^h\|_{L^2(\Omega)}.$$

It is easy to check the following estimates

$$\|\operatorname{Re} \varphi^h\|_{L^2(\Omega)} \leq \|\operatorname{Re} \varphi^h\|_{W(\Omega)}, \quad \|\operatorname{Re} f^h\|_{L^2(\Omega)} \leq c_1 \|\operatorname{Re} f\|_{L^2(\Omega)}$$

(see for details [3, Ch.3, Theorem 5]). Combining the last with (4.13), we obtain

$$(4.14) \quad \|\operatorname{Re} \varphi^h\|_{W(\Omega)} < c_1 \|\operatorname{Re} f\|_{L^2(\Omega)}.$$

Similarly we obtain the estimate

$$(4.15) \quad \|\operatorname{Im} \varphi^h\|_{W(\Omega)} < c_2 \|\operatorname{Im} f\|_{L^2(\Omega)}.$$

Finally since

$$\|\varphi^h\|_{W(\Omega)} \leq \|\operatorname{Re} \varphi^h\|_{W(\Omega)} + \|\operatorname{Im} \varphi^h\|_{W(\Omega)}$$

Estimates (4.14), (4.15) imply (4.10).  $\square$

Let us now consider the limiting process. As in [3, Ch.3], by the step function  $\varphi^h$  we construct the smooth (i.e. of the class  $C^1$ ) function  $J^h \varphi^h$ . It is convenient to take  $J^h \varphi^h$  in the form

$$J^h \varphi^h(x, y) = h^{-2} \int_x^{x+h} \int_y^{y+h} \varphi^h(\xi, \eta) d\xi d\eta.$$

This is the well known Steklov function with the averaging radius equal to the parameter  $h$  (the scale of the net). We can also define the 1-form  $J^h A^h \in \Lambda_{(1)}^1(\Omega)$  as

$$J^h A^h(x, y) = J^h A^{1h}(x, y) dx + J^h A^{2h}(x, y) dy.$$

Denote by  $\dot{W}^1(\Omega)$  the Sobolev space of complex-valued function which satisfy the homogeneous Dirichlet condition.

Let us consider some sequence  $\{h_n\}$  such that  $h_n \rightarrow 0$  as  $n \rightarrow +\infty$ . For convenient further we will write  $h$  instead  $h_n$ .

Let  $A \in \Lambda_{(1)}^1(\Omega)$  in the operator  $-\Delta_A$ . We have the statement.

**Theorem 4.2.** *Let a step function  $\varphi^h$  be the solution of the Dirichlet problem (4.8), (4.9) for the given element  $f \in L^2(\Omega)$ . Then the sequence  $\{\varphi^h\}$  strongly converges in  $L^2(\Omega)$  to the element  $\varphi \in \dot{W}^1(\Omega)$  as  $h \rightarrow 0$ , where  $\varphi$  is the generalized solution of the corresponding Dirichlet problem for Equation (4.1). At the same time the sequence  $\{J^h \varphi^h\}$  converges to  $\varphi$  in the metric  $\dot{W}^1(\Omega)$ .*

*Proof.* Based on Theorem 4.1, the proof is similar to that of Theorem 5 [3, ch.3].  $\square$

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